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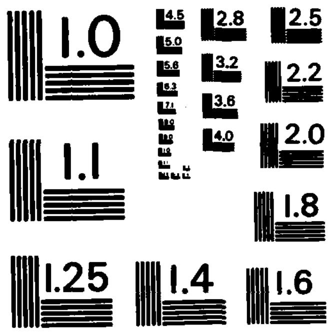

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UNSTABLE VISCO-ELASTIC FLUID FLOW EXHIBITING HYSTERITIC PHASE CHANGES

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ABSTRACT

This paper gives a mathematical theory for a visco-elastic fluid exhibiting a hysterisis loop in the shear stress vs. shear rate plane. The main rheological idea is to introduce a constitutive equation of rate type whose steady shear stress vs. shear rate locus is non-monotone. The main mathematical idea is to use a local shock structure theory to pick out the admissible solutions in loading and unloading of the applied driving force.

AMS (MOS) Subject Classifications: 76A05, 76E30.

Key Words: visco-elastic fluid f'.ow, melt fracture, ripple, phase transitions, shock, viscosity criterion.

Work Unit Number 2 - Physical Mathematics

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SIGNIFICANCE AND EXPLANATION

In processing polymers a main concern is loss of stability in what is often termed melt fracture. This loss of stability is often thought of as being due to internal material properties. This paper analyzes how a certain material model for a visco-elastic fluid yields results with some features of the "ripple" phenomena seen in melt fracture of certain polymers.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

UNSTABLE VISCO-ELASTIC FLUID FLOW EXHIBITING HYSTERITIC PHASE CHANGES

J. K. Hunter* and M. Slemrod**

0. Introduction

In his 1969 survey paper [1] on melt fracture of molten polymers J. P. Tordella described a class of experiments illustrating a phenomenon he called "ripple". In these experiments certain polymers (high density polyethylene and crystalline copolymers of tetrafluorethylene and hexofluorophylene) exhibit remarkable behavior in shear flow. While this topic has been discussed many places in the rheological literature (see [2] for a bibliography) it is only recently [3], [4] that mathematical analysis has been used in the qualitative study of the "ripple" phenomenon. In this paper we continue this program by giving a solution of a boundary value problem which has properties similar to the behavior described in [1].

First we paraphrase Tordella's observations. For shear flow in a capillary tube at sufficiently small values of stress ordinary visco-elastic flow occurs: viscosity decreases with increasing stress and the emerging stream is smooth and regular. At and above a critical stress τ_A the emerging stream has some distortion. At a second critical stress τ_B which is greater than τ_A the shear rate tends to be double valued. Furthermore

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upon increasing shear stress the shear rate possesses values on different branch of the stress vs. rate of shear curve. Decreasing stress brings a return to the original branch in a hysteretic fashion.

In this paper we attempt to give a qualitative explanation for this hysteresis effect based on an elementary visco-elastic model of rate type. The main feature of the model is that for steady flows it possesses a non-monotone shear stress vs. rate of shear constitutive relation. Such models have been suggested for phase transition-like behavior in non-linear elasticity by Ericksen [5] and more recently in non-linear visco-elasticity by Bernstein and Zapas [6]. Within the range of visco-elastic fluids such ideas seem mainly found in the Soviet literature, i.e. in the work of Vinogradov, Rutkevich and their co-workers [7], [8], [9], [10]. What makes our work different is we solve a specific boundary value problem for the steady shear flow of a visco-elastic fluid satisfying our constitutive relation. The boundary value problem (in general) possesses non-unique solutions and we use a local dynamic shock structure theory to choose a solution. This solution exhibits a hysteresis loop, double valued shear rates at critical stresses, and spatially segregated flow regimes similar to the flow birefrigence photgraphs given in [1].

1. General relations

If in a fixed Cartesian coordinate system x, y, z the velocity field of a flowing and incompressible fluid body has the form

$$v^x = 0, \; v^y = v(x,t), \; v^z = 0 \quad , \tag{1.1}$$

we say the motion is a <u>rectilinear shearing flow</u>. For such a flow the condition of incompressibility $\operatorname{div} \underline{v} = 0$ is automatically satisfied. If the fluid is a <u>simple fluid</u> in the sense of Coleman and Noll [11] then the components of the Cauchy stress obey the relations

$$\underset{\sim}{T} = -p\underset{\sim}{I} + \underset{\sim}{T}_E$$

where the extra stress $\underset{\sim}{T}_E$ satisfies

$$\begin{aligned}
T_E^{xy} = T_E^{yx} &= \underset{s=0}{\overset{\infty}{\mathfrak{z}_0}} (\Lambda_t(s)) \quad , \\
T_E^{xx} - T_E^{yy} &= \underset{s=0}{\overset{\infty}{\mathfrak{z}_1}} (\Lambda_t(s)) \quad , \\
T_E^{yy} - T_E^{zz} &= \underset{s=0}{\overset{\infty}{\mathfrak{z}_2}} (\Lambda_t(s)) \quad , \\
T_E^{zy} = T_E^{yz} &= T_E^{xz} = T_E^{zx} = 0 \quad , \\
T_E^{xx} + T_E^{yy} &+ T_E^{zz} = 0 \quad .
\end{aligned} \tag{1.2}$$

Here p is an indeterminate hydrostatic pressure and $\mathfrak{z}_0, \mathfrak{z}_1, \mathfrak{z}_2$ are real valued, generally non-linear functions of the relative shearing history $\Lambda_t(s)$,

$$\Lambda_t(s) = -\int_{t-s}^{t} v_x(x,\tau)d\tau \qquad (0 \leq s < \infty) \quad .$$

From the fluid's isotropy we know the functionals $\mathfrak{z}_0, \mathfrak{z}_1, \mathfrak{z}_2$ satisfy the relations

$$\underset{s=0}{\overset{\infty}{\mathscr{b}_0}} (-\Lambda_t(s)) = -\underset{s=0}{\overset{\infty}{\mathscr{b}_0}} (\Lambda_t(s)) \quad ,$$

$$\underset{s=0}{\overset{\infty}{\mathscr{b}_i}} (-\Lambda_t(s)) = \underset{s=0}{\overset{\infty}{\mathscr{b}_i}} (\Lambda_t(s)), \quad i = 1,2 \quad . \tag{1.3}$$

If we substitute conditions (1.2) into the equations of balance of linear momentum we find

$$0 = \frac{\partial T_E^{xx}}{\partial x} - \frac{\partial p}{\partial x} + b_1(x,t)$$

$$\rho v_t(x,t) = \frac{\partial T_E^{yx}}{\partial x} - \frac{\partial p}{\partial y} \tag{1.4}$$

$$0 = -\frac{\partial p}{\partial z}$$

where b_1 is a component of the body force $\underline{b} = (b_1,0,0)$ and T_E^{xx}, T_E^{yx} are functions of x and t only. From (1.4a) we find $\frac{\partial p}{\partial y} = \breve{\gamma}(t)$ and $b_1(x,t)$ is the body force component necessary to preserve (1.4a) as an identity. Thus $v(x,t)$ satisfies the evolution equation (1.4b).

To proceed further it is necessary to make some mathematical assumptions as to nature of the functional $\mathscr{b}_0$. For this analysis we assume $\mathscr{b}_0$ has the particularly simple form

$$T_E^{xy}(t) = \underset{s=0}{\overset{\infty}{\mathscr{b}_0}} (\Lambda_t(s)) = \int_0^\infty e^{-\lambda s}\sigma(v_x(x,t-s))ds \quad . \tag{1.5}$$

Alternatively we note that it is equivalent to (1.5) that $T_E^{xy}(t)$ be of rate type since T_E^{xy} satisfies

$$\frac{\partial}{\partial t} T_E^{xy}(t) + \lambda T_E^{xy}(t) = \sigma(v_x(\cdot,t)) \quad . \tag{1.6}$$

In the case $\breve{\gamma}$ is a constant (representing a constant applied during force) (1.6) and (1.4b) imply $v(x,t)$ satisfies the second order equation

$$\rho(v_{tt} + \lambda v_t) = \sigma(v_x)_x - \gamma \ , \tag{1.7}$$

where $\gamma = \lambda\check{\gamma}$. If we set $v_t = u$, $v_x = w$ we see that u, w satisfy the first order system

$$\begin{aligned} \rho(u_t + \lambda u) &= \sigma(w)_x - \gamma \ , \\ w_t &= u_x \ . \end{aligned} \tag{1.8}$$

We will assume our visco-elastic fluid is confined between two parallel walls of infinite extent at $x = -1$ and $x = 1$ and satisfies no-slip boundary conditions

$$v = 0 \quad \text{at} \quad x = -1,\ 1 \ . \tag{1.9}$$

(Of course this implies $u = 0$ at $x = -1, 1$.)

So far we have made no assumptions as to the nature of σ beyond the fact that (1.3) forces σ to be an odd function of its arguments. In this paper we take σ to be a smooth odd function with graph shown in Figure 1.

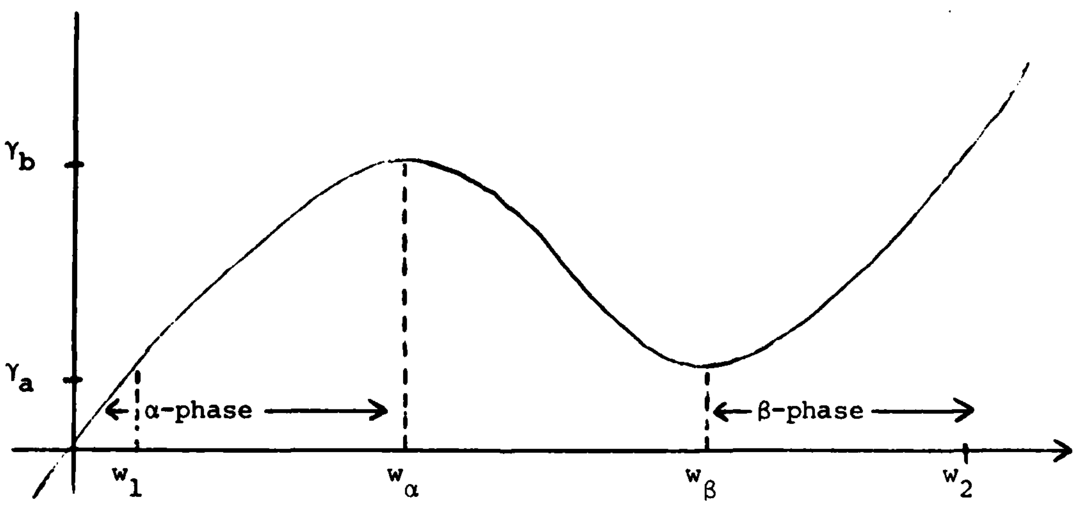


Figure 1. The graph of σ.

Here σ is such that

$$\begin{aligned} &\sigma' > 0 \quad \text{on} \quad [0,w_\alpha),\ (w_\beta,\infty) \quad , \\ &\sigma' < 0 \quad \text{on} \quad (w_\alpha,w_\beta) \quad , \\ &\sigma(w_1) = \sigma(w_\beta) = \gamma_a \quad , \\ &\sigma(w_\alpha) = \sigma(w_2) = \gamma_b \quad . \end{aligned} \tag{1.10}$$

Following the standard nomenclature of phase transitions we say that at a point where $w = v_x$ takes on values in $[0,w_\alpha)$ the fluid is in the α-phase and when w takes on values in (w_β,∞) the fluid is in the β-phase. Of course for different values of x the same fluid may be in the α and β phases simultaneously.

Since the function σ is not globally invertible we denote by σ_α^{-1}, σ_β^{-1} the respective inverses of σ in the α and β phases i.e.

$$\begin{aligned} &\sigma_\alpha^{-1}(\gamma) \in [0,w_\alpha] \quad \text{for} \quad 0 < \gamma < \gamma_b \quad , \\ &\sigma_\beta^{-1}(\gamma) \in [w_\beta,\infty) \quad \text{for} \quad \gamma_a < \gamma < \infty \quad . \end{aligned}$$

In what follows we admit solutions with values in α and β phases. However we shall not allow solutions with values in (w_α,w_β) as (1.7) is an elliptic initial value problem in this region and hence will exhibit the classical Hadamard instability.

2. Bifurcation and coalescence of fluid phases for steady motion.

In this section we will show how discontinuities between α and β phases may appear and disappear in nearly steady flow ($v_t \equiv 0$). For steady flow (1.7) implies $\sigma(v_x) = \gamma$ which upon integration yields $\sigma(v_x) = \gamma x +$ constant. By symmetry of the flow x about $x = 0$, $v(x)$ should be an even function of x and hence v_x should be odd. As noted earlier σ is an odd function by isotropy. Hence $\sigma(v_x) - \gamma x$ is odd in x. The only constant which is odd in x is zero so at equilibrium (1.7) reduces to

$$\sigma(v_x) = \gamma x \quad . \tag{2.1}$$

If $0 \leq \gamma < \gamma_a$ then $0 \leq \gamma x < \gamma_a$ for $x \in [0,1]$. In this case (2.1) has a unique solution

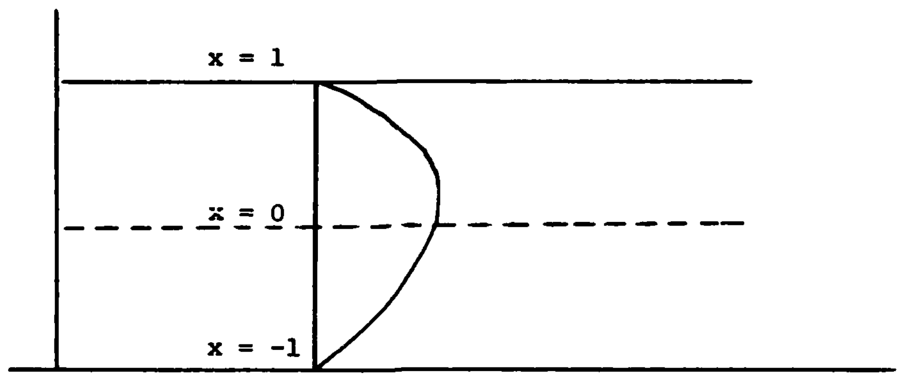
$$\begin{aligned} v(x) &= \int_1^x \sigma_\alpha^{-1}(\gamma s)ds \quad , \\ v_x(t) &= \sigma_\alpha^{-1}(\gamma x) \quad . \end{aligned} \tag{2.2}$$

On the other hand for $\gamma_a < \gamma$, the lack of a unique inverse allows (2.1) to possess non-unique solutions. For example we may have

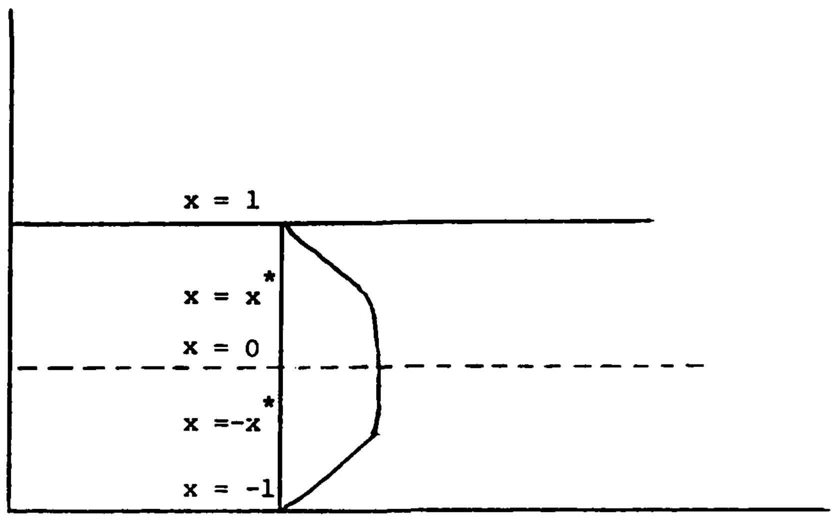
$$\begin{aligned} v(x) &= \int_{x^*}^{x} \sigma_\alpha^{-1}(\gamma s)ds + \int_1^{x^*} \sigma_\beta^{-1}(\gamma s)ds \quad , \\ v_x(x) &= \sigma_\alpha^{-1}(\gamma x) \quad , \qquad\qquad 0 \leq x \leq x^* \; ; \\ v(x) &= \int_1^x \sigma_\beta^{-1}(\gamma s)ds \quad , \\ v_x(x) &= \sigma_\beta^{-1}(\gamma x) \quad , \qquad\qquad x^* < x \leq 1 \quad . \end{aligned} \tag{2.3}$$

The function $v(x)$ given by (2.3) is continuously differentiable on $(0,x^*)$, $(x^*,1)$, continuous on $(0,1)$, and satisfies the boundary conditions $v(1) = 0$. Of course the values for $-1 \leq x < 0$ are obtained by symmetry. As x^* is an arbitrary number between γ_a/γ and 1, (2.3) represents an infinite number of solutions of (2.1) possessing a jump discontinuity in v_x

at x^*. We note that on $(0,x^*)$ the fluid is in the α-phase and on $(x^*,1)$ the fluid is in the β-phase. So the line $x = x^*$ represent a singular surface where the fluid exhibits an inter-phase jump. In addition to solutions of the form (2.3) if $\gamma_a < \gamma < \gamma_b$, (2.2) will provide a continuously differentiable solution of (2.1). Pictorially these solutions are illustrated in Figure 2.



(A)



(B)

Figure 2

(A) Solution (2.2), $0 < \gamma < \gamma_b$;

(B) Solution (2.3), $\gamma_a < \gamma$.

Finally for $\gamma > \gamma_b$ (2.1) can only possess solution existing in both α and β phases. In this a jump discontinuity in v_x at some inter-phase surface is inevitable.

Now consider the following "quasi-static" experiment. Let γ be slowly raised from zero. For $\gamma \in [0,\gamma_a)$, (2.1) possesses only C^1 solution given by (2.2). When γ is raised above γ_a we face the possibility of a spatial bifurcation occuring with the appearance of a solution of the form (2.3). Finally continuing to raise γ beyond γ_b to γ_0 guarantees the formation of a solution possessing a jump discontinuity in v_x. If we now decrease γ from γ_0 back to zero we should eventually see the α-phase solution (2.2) when γ is less than γ_a. Thus we expect to see bifurcation and coalescence of fluid phases. To understand how and where such bifurcations and coalescences take place we now appeal to a local dynamic analysis.

3. Local dynamic analysis

Our point of view is simple. We believe that for a spatial bifurcation/coalescence of fluid phases to occur near equilibrium an inter-phase singular surface across which v_x jumps must slowly propagate in the fluid. Hence if we understand when such surfaces can propagate we will know when a spatial bifurcation/coalescence of fluid phases can take place.

We shall call a C^1 surface $S : x = s(t)$ a <u>singular surface</u> for (1.8) if u and w experience jumps across S. If S is a singular surface let $(s(\bar{t}),\bar{t})$ be a fixed point on the graph of S in the $x - t$ plane and $T = \dot{s}(\bar{t})$. Denote by u_+, w_+, u_-, w_- the respective limits from the right and left as $(x,t) \to (s(\bar{t}),\bar{t})$ for u, w in the $x - t$ plane. If we put $[u] = u_+ - u_-$, $[w] = w_+ - w_-$, etc. then we know the Rankine-Huginot conditions must be satisfied, i.e.

$$\begin{aligned} \rho T[u] + [\sigma] &= 0 \quad , \\ T[w] + [u] &= 0 \quad . \end{aligned} \tag{3.1}$$

Of course (3.1) implies

$$\rho T^2 = \frac{[\sigma]}{[w]} \ . \tag{3.2}$$

Our goal is to find out which singular surfaces for (1.8) are physically meaningful. To do this we adopt the following premise: The shearing stress given by (1.5) possesses no Newtonian viscous contribution. We believe, however, that any real fluid should possess some Newtonian contribution albeit small. Hence the "good" solutions (1.7) and (1.8) should be the ones that are robust or stable (in sense to be made precise below) under small perturbations of (1.7) and (1.8) with a Newtonian viscous term. Of course this view is not original and has a long tradition dating back to Rayleigh [12]. More recent

discussions of the role of viscosity in the study of singular surfaces may be found in the books of Zel'dovich and Raizer [13] and Courant and Friedrichs [14].

If we add a linear Newtonian viscous term μv_x to the shearing stress with $\mu > 0$ denoting an (assumed) small viscosity we find (1.5) is replaced by

$$T_E^{xy}(t) = \int_0^\infty e^{-\lambda s}\sigma(v_x(x,t-s))ds + \mu v_x \quad . \tag{3.2}$$

Substituting (3.2) into the balance of linear momentum equation yields the evolution equation

$$\rho v_t^\mu = \int_0^\infty e^{-\lambda s}\sigma(v_x^\mu(x,t-s)),_x ds + \mu v_{xx}^\mu - \gamma \tag{3.3}$$

where v^μ is used to denote the solution of (3.3). If we set $w^\mu = v_x^\mu$, $u^\mu = v_t^\mu$, we readily see by differenting (3.3) with respect to t and integrating by parts that w^μ, u^μ satisfy the first order system

$$\begin{aligned} \rho(u_t^\mu + \lambda u^\mu) &= \sigma^\mu(w^\mu)_x + \mu u_{xx}^\mu - \gamma \quad , \qquad -1 < x < 1 \\ w_t^\mu &= u_x^\mu \quad , \end{aligned} \tag{3.4}$$

where $\sigma^\mu(w)$ is defined by

$$\sigma^\mu(w) = \sigma(w) + \lambda\mu w \; .$$

We now follow an argument of Dafermos [15] for the analysis of singular surfaces. Let S be a singular surface for (1.8) and $(s(\bar{t}),\bar{t})$, u_+, w_+, u_-, w_-, T as above. As we are interested in the behavior of u, w in the immediate behavior of S we introduce stretched normal and tangential coordinates

$$\xi = \frac{x-s(\bar{t})-T(t-\bar{t})}{\mu,}, \quad \eta = \frac{(T(x-s(\bar{t}))+(t-\bar{t})}{\mu} \quad .$$

Let $\bar{u}^\mu(\xi,\eta) = u^\mu(x,t)$, $\bar{w}^\mu(\xi,\eta) = w^\mu(x,t)$. Then an application of the chain rule shows that (3.4) is equivalent to

$$
\begin{aligned}
&\rho(-T\bar{u}^{\mu}_{\xi} + \bar{u}^{\mu}_{\eta} + \lambda\mu\bar{u}^{\mu}) = \\
&\qquad \sigma^{\mu\,'}(\bar{w}^{\mu})(\bar{w}^{\mu}_{\xi} + \bar{w}^{\mu}_{\eta}T) - \lambda\mu(\bar{w}^{\mu}_{\xi} + T\bar{w}^{\mu}_{\eta}) \\
&\qquad + (\bar{u}^{\mu}_{\xi\xi} + 2T\bar{u}^{\mu}_{\xi\eta} + T^{2}\bar{u}^{\mu}_{\eta\eta}) - \mu\gamma \\
&\qquad -T\bar{w}^{\mu}_{\xi} + \bar{w}^{\mu}_{\eta} = \bar{u}^{\mu}_{\xi} + T\bar{u}^{\mu}_{\eta} \quad .
\end{aligned}
\tag{3.5}
$$

A classical result of Maxwell [16] asserts that the discontinuities in u and w can occur only normal to the singular surface S. Hence if as $\mu \to 0+$ we desire u^{μ}, w^{μ} to approach the discontinuous profile of u, w, $\bar{u}^{\mu}$, $\bar{w}^{\mu}$ should tend to functions which change only in a direction normal to the singular surface. In other words we expect $\bar{u}^{\mu}, \bar{w}^{\mu} \to \bar{u}^{0}, \bar{w}^{0}$ as $\mu \to 0+$ where $\bar{u}^{0}$, $\bar{w}^{0}$ are functions of ξ alone. Thus formally taking limits as $\mu \to 0+$ in (3.5) we find

$$
\begin{aligned}
-T\rho\,\dot{\bar{u}}^{0}(\xi) &= \sigma(\bar{w}^{0})^{\cdot} + \ddot{\bar{u}}^{0}(\xi) \quad , \\
-T\,\dot{\bar{w}}^{0}(\xi) &= \bar{u}^{0}(\xi) \quad ,
\end{aligned}
\tag{3.6}
$$

where $\cdot = \frac{d}{d\xi}$. In addition for u^{μ}, w^{μ} to approximate the discontinuous profile of u, w we will need $\lim\limits_{\xi\to 0-} \bar{u}^{\mu}(\xi,0) = u_{-}$, $\lim\limits_{\xi\to 0+} \bar{u}^{\mu}(\xi,0) = u_{+}$, $\lim\limits_{\xi\to 0-} \bar{w}^{\mu}(\xi,0) = w_{-}$, $\lim\limits_{\xi\to 0+} \bar{w}^{\mu}(\xi,0) = w_{+}$. If we now formally take the limit as $\mu \to 0+$, interchanging limiting processes, we find $\bar{u}^{0}$, $\bar{w}^{0}$ should satisfy boundary conditions

$$
\begin{aligned}
\bar{u}^{0}(-\infty) &= u_{-} \ , \quad \bar{w}^{0}(-\infty) = w_{-} \quad , \\
\bar{w}^{0}(+\infty) &= u_{+} , \quad \bar{w}^{0}(+\infty) = w_{+} \quad .
\end{aligned}
\tag{3.7}
$$

This discussion motivates the following definition.

__Definition__. A singular surface $S : x = s(t)$ for (1.8) with $(s(\bar{t}),\bar{t})$, u_-, w_-, u_+, w_+, T as above is admissible according to the __Newtonian viscosity criterion__ if (3.6) possesses a C^2 solution for every point $(s(\bar{t}),\bar{t})$ on the graph S.

It will be the ability to satisfy the Newtonian viscosity criterion that will characterize the robustness or stability of the shearing stress under small perturbations of Newtonian viscous stress.

__Theorem 3.1__. As singular surface S for (1.8) is admissible according to the Newtonian viscosity criterion if and only if

$$\frac{\sigma(\hat{w}) - \sigma(w_-)}{\hat{w}-w_-} - \rho T^2 \begin{cases} > 0 & \text{if } T > 0 \\ < 0 & \text{if } T < 0 \end{cases} \tag{3.8}$$

for every $\hat{w}$ between w_- and w_+. Alternatively if $(w_+-w_-)T > 0$ (or $(w_+-w_-)T < 0$) the chord which joins $(w_-,\sigma(w_-))$ to $(w_+,\sigma(w_+))$ lies below (above) the graph of σ between w_- and w_+.

__Proof__. The proof has been given in [15] and [17]. However since it is short and straightforward we present it for completeness. First integrate (3.6) from $-\infty$ to ξ. We see that $\bar{w}^0(\xi)$ satisfies the first order ordinary differential equation

$$T^2\rho(\bar{w}^0(\xi) - w_-) - \sigma(\bar{w}^0) + \sigma(w_-) + T\dot{\bar{w}}^0(\xi) = 0 \tag{3.9}$$

with boundary conditions $\bar{w}^0(-\infty) = w_-$, $\bar{w}^0(+\infty) = w_+$. If $w_- < w_+$ a necessary and sufficient for there to exist an orbit connecting w_- to w_+ is that $T\{T^2\rho(\hat{w} - w_-) - \sigma(\hat{w}) + \sigma_-\} < 0$ for $w_- < \hat{w} < w_+$ for this gives a one directional vector field between w_- and w_+. If $w_- > w_+$ the sign changes in the inequality. But this is just a restatement of (3.8).

__Corollary 3.2__. If w_- is in the α-phase and w_+ is in the β-phase with $w_- < w_+$ then (3.8a) ((3.8b)) holding means the singular surface

propagates into the β-phase (α-phase) with the α-phase (β-phase) behind the singular surface.

Proof. This follows from the definition of ξ.

Now let us reconsider the "quasi-static" experiment in light of our admissibility criterion. We have seen that as we slowly, monotonically raise γ from zero beyond γ_a, but less than γ_b, there exists the possibility of (2.1) possessing both a single α-phase solution (2.2) or two phase solutions e.g. of the type (2.3). But for a two phase solution to form an inter-phase singular surface between α- and β-phases must appear in the original strictly α-phase solution with the β-phase propagating into the α-phase. If γ is raised slowly this inter-phase singular surface is expected to propagate slowly. So we are led to ask: when can we have an admissible slowly moving singular surface separating α- and β-phases with β-phase propagating into α-phase. According to Corollary 3.2 a slowly moving surface of this type must possess left and right limits (in the x-t plane) near w_α and w_2. See Figure 3

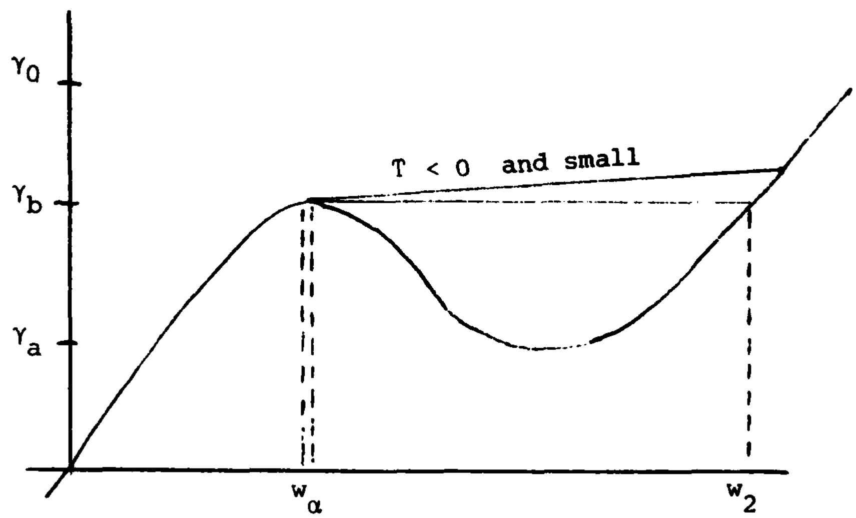


Figure 3

β-phase slowly into α-phase

What is the location of the inter-phase singular surface? If the surface were actually static with $T = 0$ and located at $x = x^* > 0$ with w possessing limits w_α and w_2 as $x \nearrow x^*$ and $x \searrow x^*$ then (2.1) implies $w_\alpha = \sigma_\alpha^{-1}(\gamma x^*)$ and $w_2 = \sigma_\beta^{-1}(\gamma x^*)$. Since $\sigma(w_\alpha) = \sigma(w_2) = \gamma_b$ we find $x^* = \gamma_b/\gamma$. Using the symmetry of v with respect to x we then see that admissible inter-phase singular surfaces form when $\gamma = \gamma_b$ at $x = \pm 1$ and propagate with β-phase moving into the channel. This is shown in Figure 4. Of course if γ increases the location of x^* asymptotically approaches $x = 0$. Also since T is not zero but merely small and negative x^* represents only an approximation to the location of the singular surface.

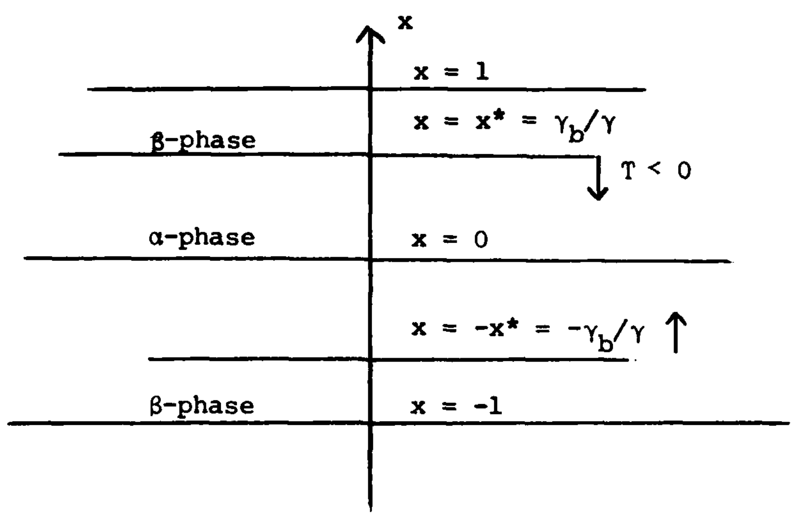


Figure 4

Solution of (2.1) for γ increasing, $\gamma > \gamma_b$.

We now move onto a second question: when can we have a slowly moving inter-phase singular surface with α-phase moving into β-phase? By an argument analogous to the one given above we see this can only occur when the limits from the left and right are near w_1 and w_β. This is illustrated in Figure 5.

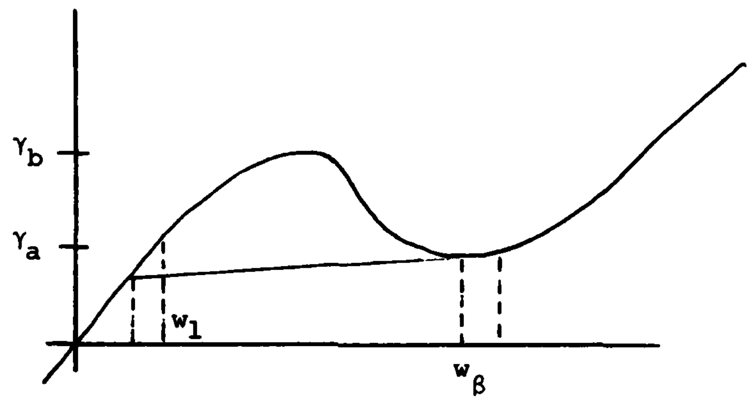


Figure 5

What is the location of this inter-phase singular surface? By an argument analogous to the one given above we find it is approximately located at $x^{**} = \gamma_a/\gamma$. We can see the effect of slowly monotonically lowering γ from some value γ_0, $\gamma_0 > \gamma_b$, in our "quasi-static" experiment. The singular surface is initally located at $x_0^* = \gamma_b/\gamma_0$ (say as shown in Figure 4 with $\gamma = \gamma_0$). As we lower γ the inter-phase singular surface cannot move until $\gamma = \frac{\gamma_a\gamma_0}{\gamma_b}$ The reason is that when $\frac{\gamma_a\gamma_0}{\gamma_b} < \gamma < \gamma_0$ if the surface (for $x > 0$) is to move upward it must occur with $1 \gg T > 0$ and α-phase moving into β-phase. But we have just observed this can only occur at $x^{**} = \gamma_a/\gamma$. However $x^{**} = x_0^*$ can only happen when $\gamma = \frac{\gamma_a\gamma_0}{\gamma_b}$ is a contradiction. On the

other hand if the surface (for $x > 0$) is to move downward we must have β-phase propagating into α-phase with $T < 0$ and small. This of course must take place at $x^* = \gamma_b/\gamma$ with increasing γ not decreasing γ. So this possibility is also excluded. So the surface remains at $x_0^* = \gamma_b/\gamma_0$ until γ reaches $\frac{\gamma_a\gamma_0}{\gamma_b}$. Thus the solution exhibits a form of shape memory in that interface "remembers" the location it was brought to via the γ increasing part of the cycle.

As we bring γ below $\frac{\gamma_a\gamma_0}{\gamma_b}$ the singular surface now begins to move slowly upward with α-phase propagating into β-phase. The surface is approximately located at $x^{**} = \gamma_a/\gamma$. When γ reaches γ_a the inter-phase singular surface moves into the channel walls and disappears. So for $0 < \gamma < \gamma_a$ we see only the α-phase solution (2.2). Our admissible solution to (2.1) is summarized in the following table.

Table:

Solution of (2.1) for γ increasing monotonically from 0 to $\gamma_0 > \gamma_b$, then decreasing monotonically to zero.

(1) γ increasing, $0 < \gamma < \gamma_b$:

$$v(x) = \int_1^x \sigma_\alpha^{-1}(\gamma s)ds, \qquad 0 < x < 1 \quad ,$$

$$v_x'(x) = \sigma_\alpha^{-1}(\gamma x) \quad ;$$

(2) γ increasing, $\gamma_b < \gamma < \gamma_0$:

$$v(x) = \int_{\gamma_b/\gamma}^{x} \sigma_\alpha^{-1}(\gamma s)ds + \int_1^{\gamma_b/\gamma} \sigma_\beta^{-1}(\gamma s)ds, \quad 0 < x < \gamma_b/\gamma ,$$

$$v_x(x) = \sigma_\alpha^{-1}(\gamma x) ,$$

$$v(x) = \int_1^x \sigma_\beta^{-1}(\gamma s)ds, \quad \frac{\gamma_b}{\gamma} < x < 1 ,$$

$$v_x(x) = \sigma_\beta^{-1}(\gamma x) ;$$

(3) γ decreasing, $\frac{\gamma_a}{\gamma_b}\gamma_0 < \gamma < \gamma_0$:

$$v(x) = \int_{\gamma_b/\gamma_0}^{x} \sigma_\alpha^{-1}(\gamma s)ds + \int_1^{\gamma_b/\gamma_0} \sigma_\beta^{-1}(\gamma s)ds, \quad 0 < x < \gamma_b/\gamma_0 ,$$

$$v_x(x) = \sigma_\alpha^{-1}(\gamma x) ,$$

$$v(x) = \int_1^x \sigma_\beta^{-1}(\gamma s)ds, \quad \frac{\gamma_b}{\gamma_0} < x < 1 ,$$

$$v_x(x) = \sigma_\beta^{-1}(\sigma x) ;$$

(4) γ decreasing, $\gamma_a < \gamma < \frac{\gamma_a}{\gamma_b}\gamma_0$:

$$v(x) = \int_{\gamma_a/\gamma}^{x} \sigma_\alpha^{-1}(\gamma s)ds + \int_1^{\gamma_a/\gamma} \sigma_\beta^{-1}(\gamma s)ds, \quad 0 < x < \gamma_a/\gamma ,$$

$$v_x(x) = \sigma_\alpha^{-1}(\gamma x) ,$$

$$v(x) = \int_1^x \sigma_\beta^{-1}(\gamma s)ds, \quad \gamma_a/\gamma < x < 1 ,$$

$$v_x(x) = \sigma_\beta^{-1}(\gamma x) ;$$

(5) γ decreasing, $0 \le \gamma \le \gamma_a$,

$$v(x) = \int_1^x \sigma_\alpha^{-1}(\gamma s)ds, \quad 0 \le x \le 1 ,$$

$$v_x(x) = \sigma_\alpha^{-1}(\gamma x) .$$

The solution for $-1 \le x < 0$ is obtained by symmetry of $v(x)$ with respect to $x = 0$.

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In Figure 6 we graph the full γ increasing, γ decreasing cycle against the boundary rate of shear $v_x(1)$. Note the occurence of a loading/unloading hysterisis loop.

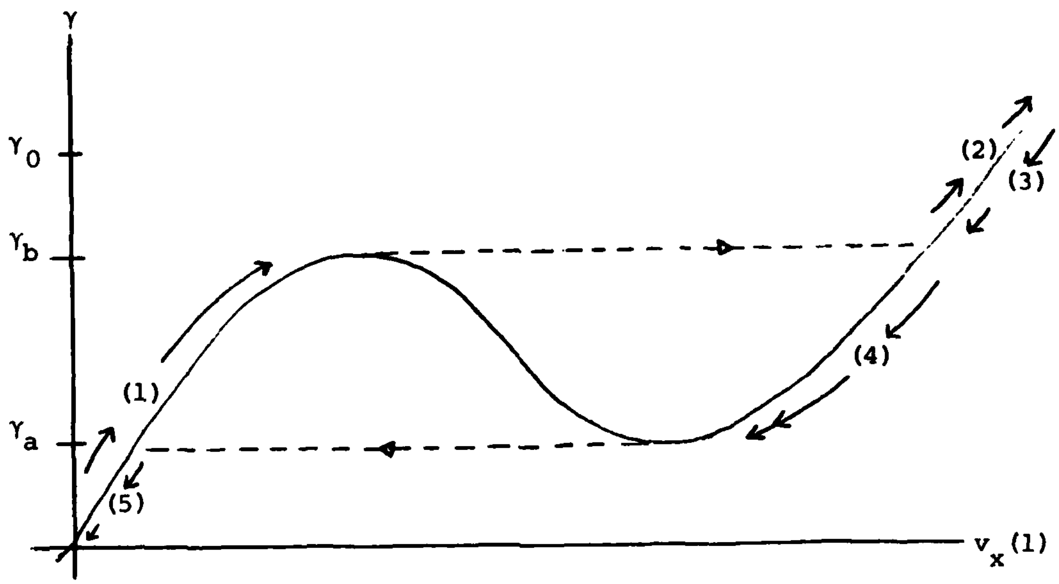

Figure 6

4. Resolution of a inter-phase discontinuity in the neighborhood of the singular surface

In the previous section we claimed that for w values in the α-phase near w_α and w values in the β-phase near w_2 there will be a slowly moving inter-phase singular surface propagating from the β-phase into α-phase. A similar claim was made for w values in the α-phase near w_1 and w values in the β-phase near w_β as to the existence of a slowly moving surface propagating from the α-phase into the β-phase. In this section we will give an argument based on local asymptotics which provides some justification for those claims.

Let us consider the first case mentioned above. When in the case when $\gamma_b < \gamma$ and slowly increasing. We assume there is a singular surface located at point $x = y_0 > 0$ at time $t = t_0$ where w_-, u_- $(w_- < w_\alpha)$ and w_+, $u_+(w_+ > w_2)$ are the limits of w, u as $x \to y_0$ from below and above respectively. As we are interested in the resolution of the discontinuity in w in an immediate neighborhood of $x = y_0$, $t = t_0$ we introduce stretched coordinates

$$T = \frac{t-t_0}{\varepsilon},\ X = \frac{x-y_0}{\varepsilon} \quad ,$$

where $\varepsilon > 0$ and small.

If we set $U(X,T;\varepsilon) = u(x,t)$, $W(X,T;\varepsilon) = w(x,t)$ we find via the chain rule that (1.8) implies

$$\begin{aligned} \rho(U_T + \lambda\varepsilon\, U) &= \sigma(W)_{,X} - \varepsilon\gamma \quad , \\ W_T &= U_X \quad . \end{aligned} \tag{4.1}$$

We look for an asymptotic expansion of U, W in ε of the form

$$U(X,T;\varepsilon) = U_0(X,T) + \varepsilon U_1(X,T) + \cdots$$

$$W(X,T;\varepsilon) = W_0(X,T) + \varepsilon W_1(X,T) + \cdots \quad .$$

Formal matching orders of ε in (4.1) shows

$$\begin{aligned} \rho U_{0_T} &= \sigma(W_0)_{,X} \\ W_{0_T} &= U_{0_X} \end{aligned} \quad . \tag{4.2}$$

If $U(X,T;\varepsilon)$ and $W(X,T;\varepsilon)$ are to approximate the behavior of u, w in a small neighborhood of (t_0,y_0) we shall need

$$\begin{aligned} U_0(X,0) &= u_-,\ W_0(X,0) = w_-,\ X < 0 \ , \\ U_0(X,0) &= u_+,\ W_0(X,0) = w_+,\ X > 0 \ . \end{aligned} \tag{4.3}$$

Thus (4.2), (4.3) represents a Riemann initial value problem for U_0, W_0.

A solution (4.2), (4.3) possessing admissible singular surfaces may be constructed in a manner similar to a method given by Lax [18] for hyperbolic conservation laws. Basically the idea is to connect u_-, w_- and u_+, w_+ to nearby values by shock or rarefaction waves. The nearby states can be then joined by a slowly moving admissible inter-phase singular surface if w_+ is near w_2, $w_+ > w_2$ and w_- is near w_α, $w_- < w_\alpha$. A similar construction can be done for w_- near w_1, $w_- < w_1$, w_+ near w_β, $w_+ > w_\beta$. These constructions may be found in the paper of James [19]. Thus for $\varepsilon_1 > 0$, sufficiently small, and

$$-\varepsilon_1 < w_- - w_\alpha < 0,\quad 0 < w_+ - w_2 < \varepsilon_1,\quad |u_+ - u_-| < \varepsilon_1$$

$$(-\varepsilon_1 < w_- - w_1 < 0,\quad 0 < w_+ - w_\beta < \varepsilon_1,\quad |u_+ - u_-| < \varepsilon_1)$$

(4.2), (4.3) possesses a solution possessing admissible shocks and an inter-phase singular surface moving with speed $T < 0$ $(T > 0)$.

We note similar constructions have been given in the forthcoming paper of M. Shearer [20].

The solution U_0, W_0 provides an asymptotic estimate of the solution of (1.8) in the neighborhood of the singular surface.

5. Discussion

We have given a theory of visco-elastic fluid flow which exhibits hysteresis. Our theory also shows the fluid exhibits a type of shape memory. The theory is qualitatively similar with respect to the hysterisis loop as the experiments described in [1]. We do not know whether the fluids described in [1] will also exhibit shape memory.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper gives a mathematical theory for a visco-elastic fluid exhibiting a hysterisis loop in the shear stress vs. shear rate plane. The main rheological idea is to introduce a constitutive equation of rate type whose steady shear stress vs. shear rate locus is non-monotone. The main mathematical idea is to use a local shock structure theory to pick out the admissible solutions in loading and unloading of the applied driving force.